Proximity Effect III

By Howard Johnson, PhD

After reading your recent newsletter on the proximity effect, I reviewed the corresponding chapter in your book. I would like to ask you for a reference dealing with the deduction of formulas 5.1 at p.190 and 5.2 at p.192.
Thank you.

- Erwin Wechsler

Thanks for your interest in High-Speed Digital Design.

Approximation [5.1] gives the current density in the solid plane underlying a trace. It says the current density at a point along the *x*-axis distance *d* removed from the trace has an intensity approximately proportional to

$$i(d) = \frac{I_0}{\pi h} \frac{1}{1+\left(d/h\right)^2} \qquad [5.1]$$

where I_0 is the (AC) current in the signal trace,
where *d* is the horizontal distance from the trace to the point under examination,
where *h* is the height of the trace above the plane, and
the dimensions are as shown in **Figure 1**.

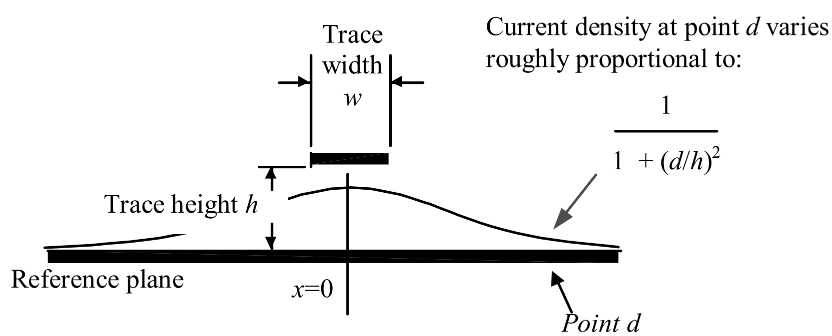

Figure 1—The distribution of high-frequency current underneath a signal trace falls off with $1/d^2$ as you move away from the trace

Approximation [5.1] is asymptotically exact only for a very small, skinny wire ($w \ll h$).

What I can do for you is show that [5.1] is the one and only correct solution to the problem of finding the distribution of returning signal current flowing on a solid plane beneath a small, skinny trace. (What I cannot do is explain *how* someone ever figured out that this was the correct solution to what will prove below to be a complicated and messy set of field equations).

The problem proceeds like this: Assume we are looking at a cross-section of an infinitely long, uniform trace and its associated solid plane (i.e., a microstrip configuration as in **Figure 1**). Position small, skinny trace at some height *h* above the plane

I will first calculate (using Ampere's law) the B-field at every point in space generated by the current flowing in the small, skinny trace.

Then I will calculate (using an integral version of Ampere's law) the B-field generated at every point in space due to the flow of current on the solid plane, assuming the current density on the plane follows equation 5.1.

Next, I will show that the vertical component of the B-field from the trace and the vertical component of the B-field from the plane cancel perfectly at the surface of the solid plane. That's the *tangential-field* boundary condition for a solid conducting plane, meaning that the magnetic field lies tangential to the plane. There is one and only one pattern of current flow on the plane that can generate this boundary condition, and that's the pattern that (at high frequencies) always occurs.

The tangential-field boundary condition is intimately related to the proximity effect. Were the perpendicular component of the magnetic field at the surface of a plane non-zero, that perpendicular magnetic field would generate eddy currents, thus shifting the distribution of current on the plane.

The only stable distribution of current is the one that generates no perpendicular component of the magnetic field, and thus no eddy currents.

Under the "good-conductor assumption" you can assume that the eddy currents will act to null out any perpendicular magnetic fields. By the way, the "good conductor assumption" just means that we are operating at a sufficiently high frequency that the resistive effects of the plane are overwhelmed by the magnetic, or eddy-current, effects.

OK, let's start by finding the B-field from the trace. First define a vector **v**, going from the trace position $(0,h)$ to some arbitrary position in space (x,y). In vector terms, we write:

$$\mathbf{v} = \begin{bmatrix} y \\ x \end{bmatrix} - \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} y-h \\ x \end{bmatrix}$$

Let the term $|\mathbf{v}|$ represent the length of vector **v**.

From Ampere's law we know the intensity of the magnetic field at point (x,y) will be equal to:

$$\left| \mathbf{Btrace}\left(\begin{bmatrix} y \\ x \end{bmatrix} \right) \right| = \frac{\mu_0 I_0}{2\pi |\mathbf{v}|}$$

The direction of the magnetic field at any point will be perpendicular to vector v. In our vector notation, a unit-length perpendicular rotation of a vector (90-degress to the right) is formed like this:

$$\mathbf{perp}\left(\begin{bmatrix} y \\ x \end{bmatrix} \right) \triangleq \begin{bmatrix} -x \\ y \end{bmatrix} \cdot \frac{1}{\sqrt{x^2 + y^2}}$$

Multiplying the direction (**perp**(**v**)) times the intensity (|**Btrace**|) gives us the final solution:

$$\mathbf{Btrace}\left(\begin{bmatrix} y \\ x \end{bmatrix} \right) = |\mathbf{Btrace}| \cdot \mathbf{perp}(\mathbf{v})$$

$$= \frac{\mu_0 I_0}{2\pi|\mathbf{v}|} \cdot \begin{bmatrix} -x \\ y-h \end{bmatrix} \cdot \frac{1}{|\mathbf{v}|}$$

$$= \frac{\mu_0 I_0}{2\pi} \cdot \begin{bmatrix} -x \\ y-h \end{bmatrix} \cdot \frac{1}{x^2+(y-h)^2}$$

We will be interested only in the vertical component (y-value) of this output of the above equation, and we only need to evaluate the y-component at locations along the horizontal axis (the reference plane). The resulting simplified, real-valued function *Btrace* has a single output (y-value only) and has a single argument (x-input position only, with the y-input assumed zero).

$$Btrace(x) = \frac{\mu_0 I_0}{2\pi} \cdot \frac{-x}{x^2+h^2}$$

Fine. The next step is to find the function Bplane(), which is generated by the current flowing in the reference plane (buckle your seat belt).

We will restrict our attention to only the y-values of this function, and only at locations along the reference plane (y =0). The resulting function is single-valued (y-value only) and has a single argument (the x-input position only, with the y-input assumed zero).

The function *Bplane(x)* is the integral of incremental contributions from every point r along the reference plane. The integrand is the incremental amount of B-field generated at point x by the current flowing within distance dr of position r. Contributions from all positions r are integrated all along the x axis.

Since point x and point r both lie along the x-axis, the incremental contribution at x from current flowing within dr of position r is naturally perpendicular to the x-axis, and has this amplitude:

$$Incremental\ \ contribution\, of\ current\, near\ position\, r = \frac{\mu_0}{2\pi}\frac{i(r)}{r-x}dr$$

The integrated contributions look like this:

$$Bplane(x) = \int_{-\infty}^{\infty} \frac{\mu_0}{2\pi}\frac{i(r)}{r-x}dr$$

NOW comes the fun part. You can't directly integrate the equation above because has a big singularity in the middle of it (at r=x). There is hope, however, as the contributions in the integral on either side of the singularity *should* cancel out if we do our work carefully. I'll start by breaking the integral into two parts, the part to the right of x, and the part to the left of x.

$$Bplane(x) = \int_{-\infty}^{x} \frac{\mu_0}{2\pi}\frac{i(r)}{r-x}dr + \int_{x}^{\infty} \frac{\mu_0}{2\pi}\frac{i(r)}{r-x}dr$$

Now on the left integral I'll make a substitution u such that x-u=r, and on the right I'll substitute u such that x+u=r.

$$Bplane(x) = \int_{\infty}^{0} \frac{\mu_0}{2\pi} \frac{i(x-u)}{-u}(-du) + \int_{0}^{\infty} \frac{\mu_0}{2\pi} \frac{i(x+u)}{u} du$$

Flip the limits on the left-hand integral, resolve the minus signs, and combine terms inside a single integration:

$$Bplane(x) = \frac{\mu_0}{2\pi} \int_0^{\infty} \frac{i(x+u)}{u} - \frac{i(x-u)}{u} du$$

Now plug in our known expressions for *i(r)*, recognizing that we should use negative current (current on the trace flows into the diagram, current on the reference plane flows back out at us – equation [5.1] merely shows the magnitude and not the direction of current).

$$Bplane(x) = \frac{\mu_0}{2\pi} \int_0^{\infty} \frac{1}{u} \frac{-I_0}{\pi h} \left(\frac{1}{1+((x+u)/h)^2} - \frac{1}{1+((x-u)/h)^2} \right) du$$

Add together the two inside terms using a least-common denominator:

$$Bplane(x) = \frac{\mu_0}{2\pi} \int_0^{\infty} \frac{1}{u} \frac{-I_0}{\pi h} \frac{\left(1+((x-u)/h)^2\right) - \left(1+((x+u)/h)^2\right)}{\left(1+((x+u)/h)^2\right)\left(1+((x-u)/h)^2\right)} du$$

Expand the quadratic terms in the numerator, and eliminate canceling terms:

$$Bplane(x) = \frac{\mu_0}{2\pi} \int_0^{\infty} \frac{1}{u} \frac{-I_0}{\pi h} \frac{1 + x^2/h^2 - 2xu/h^2 + u^2/h^2 - 1 - x^2/h^2 - 2xu/h^2 - u^2/h^2}{\left(1+((x+u)/h)^2\right)\left(1+((x-u)/h)^2\right)} du$$

$$= \frac{\mu_0}{2\pi} \int_0^{\infty} \frac{1}{u} \frac{-I_0}{\pi h} \frac{-4xu/h^2}{\left(1+((x+u)/h)^2\right)\left(1+((x-u)/h)^2\right)} du$$

If you did all the algebra correctly, you should see a *u* term in the numerator which cancels *u* term in the denominator of the integrand. That cancellation is the miracle that removes our singularity. I'll cancel the *u* terms, and also multiply top and bottom by h^4.

$$Bplane(x) = \frac{\mu_0}{2\pi} \int_0^{\infty} \frac{-I_0}{\pi h} \frac{-4xh^2}{\left(h^2+(x+u)^2\right)\left(h^2+(x-u)^2\right)} du$$

We will next make the leap of assuming that *x* can be a COMPLEX argument (with real and imaginary parts). We don't need a complex argument except that this artificial extension of the function *Bplane ()* is going to help us evaluate the integral.

Considering *x* to be complex, you can see that the integrand of *Bplane ()* is a nice, analytic function except for a total of four poles, at the following four locations:

p1: u=-x + jh
p2: u=-x – jh
p3: u= x + jh
p4: u= x – jh

I propose to evaluate the integral using the method called "calculus of residues". More information about this method, if you are not familiar with it, may be found in any book on Laplace transforms.

According to the theory of path integration, the integral we seek *Bplane ()* is precisely half as big as an equivalent integral taken over a different path. The path *Z* that I have in mind that starts at -∞ (on the real axis), heads straight toward zero, and then continues on to +∞ (also on the real axis).

Basically, path *Z* integrates the function of interest from -∞ to +∞ (instead of merely from 0 to +∞). Because the integrand isan even function (symmetrical around *u*=0), integration around path *Z* generates a result twice as large as we need. The advantage of using an integral from minus ∞ to plus ∞ is that we may now apply residue theory.

According to the residue method, the contour integral along path *Z* may be evaluated by simply summing all the residues at pole locations within the top half-plane (positive imaginary part).

The relevant poles are p1 and p3. Remember we need to incorporate the term ½ in our answer because the integral along path *Z* exceeds by a factor of two the value we seek. In the next equation I have pulled out all the constant terms so we don't have to keep seeing them in the residue calculations, and applied the residue theorem. The last factor ($j2\pi$) is part of the residue theorem.

$$Bplane(x) = \frac{1}{2}\frac{\mu_0}{2\pi}\frac{-I_0}{\pi h}\left(-4xh^2\right)\left(j2\pi\right)\left(sum\,of\;residues\right)$$

The relevant residues are defined by the integrand function, which is:

$$f(u) = \frac{1}{\left(h^2+(x+u)^2\right)\left(h^2+(x-u)^2\right)}$$

Each residue equals:

$$residue\,at\;p1 = f(p1)\cdot(u-p1)$$
$$residue\,at\;p3 = f(p3)\cdot(u-p3)$$

It will help to express f(r) in fully-factored notation:

$$f(u) = \frac{1}{(u-p1)(u-p2)(u-p3)(u-p4)}$$

(You should check the algebra here - the factorization should work out perfectly). We can now express the residues like this:

$$residue\ at\ p1 = \frac{1}{(u-p2)(u-p3)(u-p4)},\ evaluated\ at\ u = p1$$
$$= \frac{1}{(p1-p2)(p1-p3)(p1-p4)}$$

$$residue\ at\ p3 = \frac{1}{(u-p1)(u-p2)(u-p4)},\ evaluated\ at\ u = p3$$
$$= \frac{1}{(p3-p1)(p3-p2)(p3-p4)}$$

Next let's work on summing the residues:

$$sum\ of\ residues = \frac{1}{(p1-p2)(p1-p3)(p1-p4)} + \frac{1}{(p3-p1)(p3-p2)(p3-p4)}$$

Substitute the known values of *p1-4* and combine the fractions using the least common denominator:

$$sum\ of\ residues = \frac{1}{(2jh)(-2x)(-2x+2jh)} + \frac{1}{(2x)(2x+2jh)(2jh)}$$
$$= \frac{1}{(2jh)(2x)}\left(\frac{1}{2x-2jh} + \frac{1}{2x+2jh}\right)$$
$$= \frac{1}{(2jh)(2x)}\left(\frac{4x}{4x^2+4h^2}\right)$$

Plug this latest expression back into the expression for B_plane_Y() (and simplify the first terms):

$$Bplane(x) = \frac{1}{2}\frac{\mu_0}{2\pi}\frac{-I_0}{\pi h}(-4xh^2)(j2\pi)\frac{1}{(2jh)(2x)}\left(\frac{4x}{4x^2+4h^2}\right)$$
$$= \frac{\mu_0 I_0}{2\pi}\frac{x}{x^2+h^2}$$

As you can see, the expression for *Bplane* precisely cancels the expression for the *Btrace* for all values of x. Therefore, given current I_0 on the trace and the current distribution [5.1] on the plane, the perpendicular component of the magnetic field near the reference plane is zero. This proves that the current density function *i(x)* from [5.1] must have been correct.

I realize that the mathematical details obscure the basic fact that equation [5.1] just happens to describe what currents in a flat plane tend to do. There really isn't any other intuitive justification for it, except to point out couple of facts.

First, the distribution falls off as $1/x^2$. If you think about it, I'm sure you will recall that the field density from a single wire falls off as $1/x$, so the field density from a pair of currents (the signal current and its return path along the plane) must fall off one degree faster, or $1/x^2$.

Also, in the region near x=0 underneath the trace the current is fairly constant. I like that. By the way, in real life if you use a wider trace (*w* comparable to *h*) you will see a wider flat region in the middle of the distribution, with the $1/x^2$ tails falling off to each side.

Given that equation [5.1] on page 190 is true, and given that the crosstalk is proportional to the local magnetic field strength near the reference plane [5.1] times the height of the victim above the reference plane, you can derive equation [5.2] for crosstalk on page 192.

I hope these comments are helpful to you.

Best regards,
Dr. Howard Johnson

Comments welcome! hsdd@signalintegrity.com
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